Partial Differentiations

The Gradient & Directional Derivative

$f(x, y, z) = 0$

$\vec{\upsilon} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$, $\vec{i}, \vec{j}$ & $\vec{k}$ unit vectors

The **Directional Derivative** of $f(x, y, z)$ at $p_0(x_0, y_0, z_0)$ in the direction of $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\Rightarrow$ $D.D. = \vec{\upsilon} \cdot \vec{u}$, $\vec{u} = \frac{\vec{A}}{|\vec{A}|}$ & $\vec{\upsilon}$ gradient

Ex.1:

Find D.D. of $f = x^3 - xy^2 - z$ at (1, 1, 0) in the direction of $\vec{A} = 2\vec{i} - 3\vec{j} + 6\vec{k}$

Sol.:

First we find $\vec{\upsilon}$

$\frac{\partial f}{\partial x} = 3x^2 - y^2 = 3(1) - (1) = \mathbf{2}$

$\frac{\partial f}{\partial y} = -2xy \quad = -2(1)(1) = \mathbf{-2}$

$\frac{\partial f}{\partial z} = \mathbf{-}1$

$\therefore \ \vec{\upsilon} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$\therefore \ \vec{\upsilon} = 2\vec{i} + -2\vec{j} - \vec{k}$

$\Rightarrow$ $D.D. = \vec{\upsilon} \cdot \vec{u}$,

$= \vec{\upsilon} \cdot \frac{\vec{A}}{|\vec{A}|}$

$= (2\vec{i} + -2\vec{j} - \vec{k}) \cdot \frac{2\vec{i} + -3\vec{j} + 6\vec{k}}{\sqrt{4 + 9 + 36}}$, $\vec{i} \cdot \vec{i} = 1$

$= \frac{4 + 6 - 6}{\sqrt{49}} = \frac{4}{7}$

Maxima, Minima & Saddle point

$f(x,y)$ will have M , m, S according to:

1) $f_x = 0$, $f_y = 0$ to find the suggested point (a,b).

2) $f_{xx} \cdot f_{yy} - (f_{xy})^2 \succ 0$

Then (a,b) is **M** or **m** according to f_{xx} negative or positive.

3) $f_{xx} \cdot f_{yy} - (f_{xy})^2 \prec 0$

Then (a,b) is a saddle point.

4) $f_{xx} \cdot f_{yy} - (f_{xy})^2 = 0$

<u>Ex.1:</u> Locate M,m & S (if any)

$$f = x^2 - xy + y^2 + 2x + 2y - 4$$

$$f_x = 2x - y + 2$$

$$f_y = -x + 2y + 2$$

$$2x - y + 2 = 0 \quad ...(1)$$

$$-x + 2y + 2 = 0 \quad ...(2)$$

multi (1) by 2 + (2)

$$\Rightarrow \quad 3x + 6 = 0 \quad \Rightarrow \quad x = -2 \quad , \quad y = -2 \quad , \quad (-2,-2)$$

$$f_{xx} = 2 \quad , \quad f_{yy} = 2 \quad , \quad f_{xy} = -1$$

$$f_{xx} \cdot f_{yy} - (f_{xy})^2 = (2)(2) - 1 = 3 \quad \succ 0$$

Since $f_{xx} \succ 0 \quad \Rightarrow \quad (-2,-2)$ is **m**

<u>Ex.2:</u> Locate M,m & S (if any)

$$f = x^3 + y^3 - 3axy$$

$f_x = 3x^2 - 3ay$
$f_y = 3y^2 - 3ax$

$3x^2 - 3ay = 0 \quad ...(1)$
$3y^2 - 3ax = 0 \quad ...(2)$

From (1) $\Rightarrow \quad y = \dfrac{x^2}{a}$

In (2) $\Rightarrow \quad \dfrac{x^4}{a^2} - ax = 0$

$\Rightarrow \quad x^4 - a^3 x = 0$
$x(x^3 - a^3) = 0$
$\Rightarrow \quad x = 0 \quad , \quad x = a$
$\therefore \; y = 0 \quad , \quad y = a$

$\Rightarrow \quad (0,0) \;\&\; (a,a)$

$f_{xx} = 6x \quad , \quad f_{yy} = 6y \quad , \quad f_{xy} = -3a$

1) $at \;\; (0,0) \;\; \Rightarrow \;\; f_{xx} = 0 \quad , \quad f_{yy} = 0 \quad , \quad f_{xy} = -3a$
$f_{xx} \cdot f_{yy} - (f_{xy})^2 = -9a^2 \quad \prec 0$

(0,0) is a saddle point

2) $at \;\; (a,a) \;\; \Rightarrow \;\; f_{xx} = 6a \quad , \quad f_{yy} = 6a \quad , \quad f_{xy} = -3a$
$f_{xx} \cdot f_{yy} - (f_{xy})^2 = (6a)(6a) - 9a^2$
$= 36a^2 - 9a^2 = 27a^2 \quad \succ 0$

i) if $a \succ 0 \;\; \Rightarrow \;\; f_{xx} \succ 0 \;\; \Rightarrow \;\; (a,a)$ is **m**
ii) if $a \prec 0 \;\; \Rightarrow \;\; f_{xx} \prec 0 \;\; \Rightarrow \;\; (a,a)$ is **M**

Problems:

Find $\dfrac{\partial f}{\partial x} \; , \; \dfrac{\partial f}{\partial y} \; , \; \dfrac{\partial f}{\partial z}$

1) $f(x,y,z) = z \sin^{-1} \dfrac{y}{x}$

2) $f(x,y,z) = \dfrac{x(2 - \cos 2y)}{x^2 + y^2}$

3) Find $\dfrac{\partial\omega}{\partial\upsilon}$ when

$u=0$, $\upsilon=0$ if $\omega = x^2+\dfrac{y}{x}$, $x=u-2\upsilon+1$, $y=2u+\upsilon-2$

4) If $\omega = f\left(\dfrac{xy}{x^2+y^2}\right)$, show that $x\dfrac{\partial\omega}{\partial x}+y\dfrac{\partial\omega}{\partial y}=0$

5) If $\omega = f(x,y)$, and $x=r\cos\theta$, $y=r\sin\theta$, show that

$$(\frac{\partial\omega}{\partial x})^2+\frac{1}{r^2}(\frac{\partial\omega}{\partial y})^2 = f_x^2+f_y^2$$

6) If $f(x,y,z)=0$ & $z=x+y$, find $\dfrac{dz}{dx}$

7) Find the directional derivative of $f(x,y)=x\tan^{-1}\dfrac{y}{x}$ at (1,1) in the direction of $\vec{A}=2\vec{i}-\vec{j}$

8) In which direction is the directional derivative of $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$

9) The D.D. of $f(x,y)$ at $p_0(1,2)$ in the direction towards $p_1(2,3)$ is $2\sqrt{2}$ and the D.D. at $p_0(1,2)$ towards $p_2(1,0)$ is -3 , find D.D. at p_0 towards the origin.

References:

1- calculus & Analytic Geometry (Thomas).
2- Calculus (Haward Anton).
3- Advanced Mathematics for Engineering Studies **(أ. رياض احمد عزت)**